

THU-95/31
hep-th/9601044

APPLICATIONS OF SPECIAL GEOMETRY[†]

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ABSTRACT

We review characteristic features of $N = 2$ supersymmetric vector multiplets and discuss symplectic reparametrizations and their relevance for monopoles and dyons. We close with an analysis of perturbative corrections to the low-energy effective action of $N = 2$ heterotic superstring vacua.

[†] Invited talk given at the Workshop on Strings, Gravity & Related Phenomena, Trieste, 29-30 June 1995; to be published in the proceedings.

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ABSTRACT

We review characteristic features of $N = 2$ supersymmetric vector multiplets and discuss symplectic reparametrizations and their relevance for monopoles and dyons. We close with an analysis of perturbative corrections to the low-energy effective action of $N = 2$ heterotic superstring vacua.

1. Introduction

Special geometry refers to the target-space geometry of $N = 2$ supersymmetric vector multiplets, possibly coupled to supergravity [1]. The physical states of a vector multiplet are described by gauge fields W_μ^I , doublets of Majorana spinors Ω_i^I and complex scalars X^I . The kinetic term for the scalars is a nonlinear sigma model which defines the metric of the target space, the space parametrized by the scalar fields.

The characteristic features of special geometry are as follows. The Lagrangian is encoded in a holomorphic prepotential $F(X)$. In rigid supersymmetry the fields X^I can be regarded as independent coordinates ($I = A = 1, \dots, n$). In the local case there is one extra vector multiplet labeled by $I = 0$, which provides the graviphoton, but now the $n + 1$ fields X^I are parametrized in terms of n holomorphic coordinates z^A . Here one often makes use of so-called *special* coordinates defined by $z^A = X^A/X^0$. The target space is Kählerian and the Kähler potential is given by (the subscripts on F denote differentiation)

$$\begin{aligned} K(X, \bar{X}) &= -i\bar{X}^A F_A(X) + iX^A \bar{F}_A(\bar{X}), \\ K(z, \bar{z}) &= -\log[-i\bar{X}^I(\bar{z})F_I(z) + iX^I(z)\bar{F}_I(\bar{z})], \end{aligned} \quad (1)$$

for rigid and for local supersymmetry, respectively. In the latter case, the $2n + 2$ quantities (X^I, F_I) are parametrized by n complex coordinates z^A . In a more mathematical context they are regarded as holomorphic sections, defined projectively, of a flat $Sp(2n + 2, \mathbf{R})$ vector bundle [2, 3]. The ensuing metric satisfies the following curvature relations

$$\begin{aligned} R^A_{BC}{}^D &= -\mathcal{W}_{BCE}\bar{\mathcal{W}}^{EAD}, \\ R^A_{BC}{}^D &= 2\delta_{(B}^A\delta_{C)}^D - e^{2K}\mathcal{W}_{BCE}\bar{\mathcal{W}}^{EAD}, \end{aligned} \quad (2)$$

respectively, for the two cases. Here the tensor \mathcal{W}_{ABC} is related to the third derivative of $F(X)$; for global supersymmetry one has $\mathcal{W}_{ABC} = iF_{ABC}$, while for local

supersymmetry the relevant expression reads

$$\mathcal{W}_{ABC} = iF_{IJK}(X(z)) \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial z^B} \frac{\partial X^K(z)}{\partial z^C}. \quad (3)$$

The kinetic terms for the super-Yang-Mills Lagrangian take the form

$$\begin{aligned} \mathcal{L} = & \frac{i}{4\pi} \left(D_\mu F_I D^\mu \bar{X}^I - D_\mu X^I D^\mu \bar{F}_I \right) - \frac{1}{8\pi} \text{Im} F_{IJ} (\bar{\Omega}^{iI} \overleftrightarrow{\not{D}} \Omega_i^I) \\ & - \frac{i}{16\pi} \left(\mathcal{N}_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu J} - \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{-I} F^{-\mu\nu J} \right), \end{aligned} \quad (4)$$

where Ω_i and Ω^i are the chiral components of a Majorana spinor doublet, $F_{\mu\nu}^{\pm I}$ denote the selfdual and anti-selfdual field-strength components, and¹

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im}(F_{IK}) \text{Im}(F_{JL}) X^K X^L}{\text{Im}(F_{KL}) X^K X^L}. \quad (5)$$

Other terms of the Lagrangian are (we suppress the auxiliary fields which may contribute in the presence of hypermultiplets),

$$\begin{aligned} & -\frac{i}{16\pi} \left(\bar{\Omega}_i^I (\not{D} F_{IJ}) \Omega^{iJ} + \bar{\Omega}_i^I (\not{D} \bar{F}_{IJ}) \Omega^{iJ} \right) \\ & + \frac{i}{32\pi} \left(F_{IJK} \bar{\Omega}_i^I \sigma \cdot F^{-J} \Omega_j^K \varepsilon^{ij} - \bar{F}_{IJK} \bar{\Omega}^{iI} \sigma \cdot F^{+J} \Omega_j^K \varepsilon_{ij} \right) \\ & + \frac{1}{2\pi} \text{Im} F_{IJ} \left(-\frac{1}{2} g f_{KL}^J [\bar{\Omega}_i^I \bar{X}^K \Omega_j^L \varepsilon^{ij} + \bar{\Omega}^{iI} X^K \Omega_j^L \varepsilon_{ij}] \right. \\ & \quad \left. + g^2 (f_{KL}^I \bar{X}^K X^L) (f_{MN}^J \bar{X}^M X^N) \right), \end{aligned} \quad (6)$$

where the f_{JK}^I are the structure constants of the gauge group.

Special geometry is relevant for the moduli of Calabi-Yau three-folds. This intriguing connection can be understood in the context of type-II superstrings, whose compactification on Calabi-Yau manifolds leads to four-dimensional effective actions with local $N = 2$ supersymmetry. In these field theories the sigma-model fields have no potential. Therefore their vacuum-expectation values are undetermined and parametrize the (classical) ground states of this field theory, up to certain equivalence transformations. The metric of the non-linear sigma model (in four space-time dimensions) is therefore related to the metric on the moduli space of superstring ground states associated with Calabi-Yau spaces [4]. Therefore this moduli space must exhibit special geometry. For the complex-structure moduli, the periods of the $(3,0)$ form correspond to the sections (X^I, F_J) , which transform under symplectic rotations induced by changes in the corresponding homology basis [2, 5]. These symplectic transformations exist already at the level of the supersymmetric field theory, as we discuss below.

¹In the rigid case, \mathcal{N} consists of only the first term and the $I = 0$ component is suppressed. In general, \mathcal{N} is complex. Its imaginary part is related to the gauge coupling constant, its real part to a generalization of the θ angle.

2. Symplectic reparametrizations

From the Lagrangian (4) one defines the tensors

$$G_{\mu\nu I}^+ = \mathcal{N}_{IJ} F_{\mu\nu}^{+J}, \quad G_{\mu\nu I}^- = \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{-J}, \quad (7)$$

so that the Bianchi identities and equations of motion for the Abelian gauge fields can be written as

$$\partial^\mu (F_{\mu\nu}^{+I} - F_{\mu\nu}^{-I}) = 0, \quad \partial^\mu (G_{\mu\nu I}^+ - G_{\mu\nu I}^-) = 0. \quad (8)$$

These are invariant under the transformation

$$\begin{pmatrix} F_{\mu\nu}^{+I} \\ G_{\mu\nu I}^+ \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{+I} \\ G_{\mu\nu I}^+ \end{pmatrix}, \quad (9)$$

where U_J^I , V_I^J , W_{IJ} and Z^{IJ} are constant real $(n+1) \times (n+1)$ submatrices. The transformations for the anti-selfdual tensors follow by complex conjugation. From (7) and (9) one derives that \mathcal{N} transforms as

$$\mathcal{N}_{IJ} \longrightarrow (V_I^K \mathcal{N}_{KL} + W_{IL}) [(U + Z\mathcal{N})^{-1}]^L_J. \quad (10)$$

To ensure that \mathcal{N} remains a symmetric tensor, at least in the generic case, the transformation (9) must be an element of $Sp(2n+2, \mathbf{R})$ (we disregard a uniform scale transformation). The required change of \mathcal{N} is induced by a change of the scalar fields, implied by

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \quad (11)$$

In this transformation we include a change of F_I . Because the transformation belongs to $Sp(2n+2, \mathbf{R})$, one can show that the new quantities \tilde{F}_I can be written as the derivatives of a new function $\tilde{F}(\tilde{X})$. The new but equivalent set of equations of motion one obtains by means of the symplectic transformation (properly extended to other fields), follows from the Lagrangian based on \tilde{F} . In special cases F remains unchanged, $\tilde{F}(\tilde{X}) = F(\tilde{X})$, so that the theory is *invariant* under the corresponding transformations.

The symplectic transformations act on the (anti-)selfdual components of the field strengths (cf. (9)). Such duality transformations are known to appear in all extended supergravity theories in four space-time dimensions [6] and were studied extensively as continuous invariances of the equations of motions (for $N = 2$, we refer to [1, 7, 8]). For instance, the class of functions

$$F(X) = d_{ABC} \frac{X^A X^B X^C}{X^0}, \quad (12)$$

generally leads to symplectic invariances associated with finite parameters b^A and defined by

$$U(b) = V^T(-b) = \begin{pmatrix} 1 & 0 \\ b^A & \mathbf{1}_n \end{pmatrix}, \quad W(b) = \begin{pmatrix} -(d b b b) & -3(d b b)_B \\ 3(d b b)_A & 6(d b)_{AB} \end{pmatrix}, \quad Z(b) = 0. \quad (13)$$

In terms of the special coordinates, these give rise to

$$z^A \longrightarrow z^A + b^A. \quad (14)$$

Symplectic transformations with $Z = 0$ can always be realized on the vector potentials and thus leave the Lagrangian invariant (possibly up to a total divergence corresponding to a shift in the θ angles).

Later it was realized that symplectic transformations can be used to relate different functions $F(X)$ describing equivalent equations of motion [9]. These reparametrizations were exploited in the context of Calabi-Yau manifolds and, more recently, for $N = 2$ nonabelian gauge theories to describe singularities in the Wilsonian action that originate from the emergence of massless states at strong coupling in terms of a dual theory at weak coupling [10]. Generically duality transformations mix electric and magnetic fields. The interchange of electric and magnetic fields is known as electric-magnetic duality. For instance, for $U = V = 0$ and $W = -Z = \mathbf{1}$, $F_{\mu\nu}^{+I}$ and $G_{\mu\nu I}^+$ are simply interchanged, while \mathcal{N} transforms into $-\mathcal{N}^{-1}$. Since the coupling constants are thus replaced by their inverses, electric-magnetic duality relates the strong- and weak-coupling description of the theory. Electric-magnetic duality is a special case of so-called S duality. The coupling constant inversion is part of an $SL(2, \mathbf{Z})$ group. This duality is also known in the context of string theory [11] and lattice gauge theories [12]. Other symplectic transformations induce a shift of the generalized θ angles. In nonabelian gauge theories θ is periodic, so that \mathcal{N} is defined up to the addition of certain *discrete* real constants. This is a generic feature, both here as well as in string theory; the nonperturbative dynamics restricts the symplectic transformations to a discrete subgroup. For an earlier account of confinement phases in nonabelian gauge theories where duality transformations were important, see [13].

3. Semiclassical consequences of monopoles and dyons

To elucidate some important features of the symplectic reparametrizations, let us discuss the effective action of abelian gauge fields, possibly obtained from a nonabelian theory by integrating out certain fields. We write the matrix \mathcal{N} in terms of generalized coupling constants and θ angles, according to

$$\mathcal{N}_{IJ} = \frac{\theta_{IJ}}{2\pi} - i \frac{4\pi}{g_{IJ}^2}. \quad (15)$$

This matrix can be viewed as a generalization of the permeability and permittivity that is conventionally used in the treatment of electromagnetic fields in the presence of a medium. The fields $G_{\mu\nu I}$ are thus generalizations of the displacement and magnetic fields, while $F_{\mu\nu}^I$ corresponds to the electric fields and magnetic inductions. So far we have considered an abelian theory without charges. It is straightforward to introduce electric charges by including an electric current in the Lagrangian. To consider duality transformations one must also include magnetic currents into the field equations, so

that when electric fields transform into magnetic fields and vice versa, the electric and magnetic currents transform accordingly. These magnetic currents occur as sources in the Bianchi identity and describe magnetic monopoles.

Electric and magnetic charges are conveniently defined in terms of flux integrals over closed spatial surfaces that surround the charged objects,

$$\begin{aligned}\oint_{\partial V} (F^+ + F^-)^I &= 2\pi q_m^I, \\ \oint_{\partial V} (G^+ + G^-)_I &= -2\pi q_{eI}.\end{aligned}\tag{16}$$

With these definitions a static point charge at the origin exhibits magnetic inductions and electric fields equal to $\vec{r}/(4\pi r^3)$ times $2\pi q_m^I$ and $\frac{1}{2}g^2(q_{eI} + q_m^J \theta_{IJ}/2\pi)$, respectively. Note that q_e does not coincide with the electric charge. The θ -dependent mixing of the electric and magnetic charges was first noted in [14] and follows directly from the generalized Maxwell equations in the presence of the θ angle [15, 13]. From (16) it follows that the charges must transform under symplectic rotations according to

$$\begin{pmatrix} q_m^I \\ -q_{eI} \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} q_m^I \\ -q_{eI} \end{pmatrix}.\tag{17}$$

As is well known, the charges are subject to a generalized Dirac quantization condition, due to Schwinger and Zwanziger [16], according to which $q_e q'_m - q_m q'_e$ must be a multiple of $2\hbar$. This implies that the allowed electric and magnetic charges comprise a lattice such that surface elements spanned by the lattice vectors are equal to a multiple of the Dirac unit $2\hbar$. In addition, this lattice should be consistent with the periodicity of the θ angle², $\theta \rightarrow \theta + 2\pi$, which corresponds to $\mathcal{N} \rightarrow \mathcal{N} + \mathbf{1}$. This shift is associated with a symplectic transformation with $U = V = W = \mathbf{1}$ and $Z = 0$, so that the charges transform as $q_e \rightarrow q_e - q_m$ and $q_m \rightarrow q_m$. This transformation must be contained in the discrete subgroup $Sp(2n + 2, \mathbf{Z})$ that leaves the charge lattice invariant.

As observed by Olive and Witten [17], q_e and q_m emerge as surface integrals in the supersymmetry algebra. We derive this in a slightly more general setting for a supersymmetric Yang-Mills theory based on a holomorphic function $F(X)$. In that case the supercurrent reads

$$J_{\mu i} = \frac{1}{2\pi} \text{Im } F_{IJ} \left\{ \not{D} \bar{X}^I \gamma_\mu \Omega_i^J - \varepsilon_{ij} \left[\frac{1}{2} \sigma \cdot F^{-I} - g f_{MN}^I \bar{X}^M X^N \right] \gamma_\mu \Omega^{jJ} \right\}.\tag{18}$$

In the abelian limit one can show that this result remains the same under symplectic reparametrizations. This is not surprising in view of the fact that the symplectic reparametrizations are also applicable in a supergravity background [1]. For a more general discussion of the effect of supergravity and chiral backgrounds, see [18].

²The normalization of the θ angle is fixed by the assumption that instantons yield an integer value for the Pontryagin index $(32\pi^2)^{-1} \int d^4x {}^*FF$ in a nonabelian extension of the theory.

From the Dirac brackets (suppressing explicit spinor indices)

$$\begin{aligned}\{\Omega_j^I(x), \bar{\Omega}^{iJ}(y)\}_{x_0=y_0} &= 4\pi\hbar [(\text{Im } F)^{-1}]^{IJ} \delta_j^i \left(\frac{1+\gamma_5}{2}\gamma_4\right) \delta^3(\vec{x}-\vec{y}), \\ \{\Omega^{iI}(x), \bar{\Omega}_j^J(y)\}_{x_0=y_0} &= 4\pi\hbar [(\text{Im } F)^{-1}]^{IJ} \delta_j^i \left(\frac{1-\gamma_5}{2}\gamma_4\right) \delta^3(\vec{x}-\vec{y}),\end{aligned}\quad (19)$$

we immediately determine the anticommutators of the supersymmetry charges $Q_i \equiv \int d^3x J_i^0$, at least as far as their bosonic contributions are concerned. The first one is (a, b, c label space coordinates)

$$\{Q_i, \bar{Q}^j\} = i\hbar \delta_j^i (1 - \gamma_5) \int d^3x \left\{ \gamma_\mu T^{\mu 0} + \frac{1}{8\pi} \gamma_a \varepsilon^{abc} \partial_b \left(F_I \vec{D}_c \bar{X}^I - X^I \vec{D}_c \bar{F}_I \right) \right\}, \quad (20)$$

where $T^{\mu\nu}$ is the energy-momentum tensor (which in the abelian limit also preserves its form under symplectic reparametrizations) and the total divergence leads to a surface term which can be dropped. The second anticommutator can be written as

$$\begin{aligned}\{Q_i, \bar{Q}_j\} &= -\frac{i\hbar}{4\pi} (1 - \gamma_5) \varepsilon_{ij} \int d^3x \left\{ \varepsilon^{abc} \left[(D_a \bar{X}^I) (G^+ + G^-)_{Ibc} - (D_a \bar{F}_I) (F^+ + F^-)^I_{bc} \right] \right. \\ &\quad \left. + 4 \text{Im } F_{IJ} (D_0 \bar{X}^I) f_{KL}^J \bar{X}^K X^L \right\}.\end{aligned}\quad (21)$$

Using the equations of motion and (16), this yields the following expression for the central charge,

$$\{Q_i, \bar{Q}_j\} = i\hbar (1 - \gamma_5) \varepsilon_{ij} \left\{ \bar{X}^I q_{eI} + \bar{F}_I q_{\text{m}}^I \right\}, \quad (22)$$

where \bar{X}^I and \bar{F}_I represent the constant values taken by these quantities at spatial infinity. The right-hand side manifestly preserves its form under symplectic transformations, precisely as expected. This representation of the central charge plays an important role in the analysis of [10].

4. $N = 2$ Heterotic vacua

In heterotic string vacua the $N = 2$ space-time supersymmetry charges reside entirely in the right-moving sector. This sector decomposes into a $c = 3$ and a $c = 6$ superconformal field theory with $N = 2$ and 4 world-sheet supersymmetry, respectively. The massless spectrum that emerges from this sector together with the four-dimensional space-time sector yields the graviton, an antisymmetric tensor, the dilaton and two abelian boson fields. Together with corresponding fermions, they constitute the supergravity multiplet together with a so-called vector-tensor multiplet³ [19]. On shell the tensor field can be converted into a scalar which combines with the dilaton into a complex scalar S . The latter then belongs to an $N = 2$ vector multiplet. Other vector

³This multiplet was first considered in [20] in a study of off-shell representations of supersymmetric Yang-Mills theory. The multiplet has an off-shell central charge, which couples to a gauge field in supergravity [21].

multiplets originate from the left-moving sector. The simplest case corresponds to the toroidal compactifications from the six-dimensional $N = 1$ heterotic theory. In that case one has two corresponding moduli, denoted by T and U . The toroidal compactifications can be continuously deformed by nontrivial Wilson lines in the gauge group associated with the gauge fields accompanying the two torus periods; those deformations give rise to extra vector multiplets. Our arguments below are not restricted to toroidal compactifications, however, and we assume an arbitrary number of moduli. For definiteness we take this number to exceed 2, so that we will be dealing with at least 3 vector multiplets.

Classically the effective action that describes the vector multiplets relevant for $N = 2$ heterotic vacua is restricted by the fact that the dilaton (the real part of S) couples universally while its imaginary part acts as a generalized θ angle and must be invariant under constants shifts. This uniquely determines the holomorphic homogeneous function [22] (up to symplectic reparametrizations),

$$F_{\text{class}}(X) = -\frac{X^1}{X^0} \left[X^2 X^3 - \sum_{I \geq 4}^n (X^I)^2 \right], \quad (23)$$

which corresponds to the product manifold $[SU(1, 1)/U(1)] \times [SO(2, n-1)/(SO(2) \times SO(n-1))]$. The $SU(1, 1)/U(1)$ coordinate is the dilaton field $iS = X^1/X^0$, whose real part corresponds the string coupling constant. Other moduli are given by $iT = X^2/X^0$, $iU = X^3/X^0$, etc.; these moduli transform under the target-space duality group $SO(2, n-1)$.

The objective is to consider the perturbative string corrections to (23). For these corrections the dilaton field S acts as a loop-counting parameter, so that n -loop corrections will be inversely proportional to $(X^1)^{n-1}$. Perturbatively the effective action must be invariant under continuous shifts of the imaginary part of S (proportional to the θ angle). At the same time the dependence on S in the function $F(X)$ should remain holomorphic to all orders in perturbation theory. These two requirements restrict the possible additions to (23) to be independent of X^1 , so that there are no perturbative corrections beyond one-loop. In addition there are nonperturbative corrections, which are not covered by the above argument as they are only invariant under discrete 2π -shifts of the θ angle.

The one-loop corrections should preserve the invariance under target-space duality. However, a subtlety may occur as we expect the corrections to exhibit a certain lack of single-valuedness due to the presence of singular points in the moduli space where massive string states become massless. One may wonder whether there is an appropriate symplectic basis for the periods (X^I, F_I) in which to address these questions. The basis defined by (23) has two conspicuous features. First of all, the gauge couplings do not all become weak in the large dilaton limit, so that this does not seem a good starting point for setting up consistent string perturbation theory. Secondly, the $SO(2, n-1)$ invariance is realized by duality transformations, so that the equations of motion, but not the classical Lagrangian, are left invariant. These duality

transformations involve inversions of the gauge couplings and it is therefore plausible that these two features are related. Indeed it is possible to redefine the periods by means of a symplectic reparametrization, such that all gauge couplings vanish uniformly in the large-dilaton limit and, at the same time, the classical Lagrangian is strictly invariant under $SO(2, n-1)$. The new periods are defined by⁴

$$\hat{X}^I = (X^0, F_1, X^2, \dots, X^n), \quad \hat{F}_I = (F_0, -X^1, F_2, \dots, F_n). \quad (24)$$

Now $SO(2, n-1)$ acts linearly on both \hat{X}^I and \hat{F}_I separately,

$$\hat{X}^I \rightarrow \hat{U}^I_J \hat{X}^J, \quad \hat{F}_I \rightarrow \hat{V}_I^J \hat{F}_J, \quad (25)$$

where $\hat{V} = (\hat{U}^T)^{-1}$ and \hat{U} is an $SO(2, n-1)$ matrix. The above transformations pertain to the classical case (23) and leave the corresponding Lagrangian invariant. According to (25) X^0, X^2, \dots, X^n transform among themselves, but in a nonlinear fashion.

Can these transformations be modified when one-loop corrections are included and $F(X)$ is no longer single-valued? The answer to this question follows from the observation that adding a one-loop correction $F_{1\text{-loop}}(X)$ (which is X^1 -independent) to $F_{\text{class}}(X)$ does not affect the definitions of the \hat{X}^I . Furthermore the transformations of X^0, X^2, \dots, X^n , and therefore those of the \hat{X}^I , should remain the same at the quantum level, as these fields have fixed relations to their string vertex operators. This is not so for X^1 , which is obtained by conversion of a vector-tensor multiplet. On the other hand, the target-space duality transformations should still take a symplectic form. Therefore the modifications are restricted to

$$\hat{X}^I \rightarrow \hat{U}^I_J \hat{X}^J, \quad \hat{F}_I \rightarrow \hat{V}_I^J \hat{F}_J + \hat{W}_{IJ} \hat{X}^J, \quad (26)$$

where $\hat{V} = (\hat{U}^T)^{-1}$ and $\hat{W} = \hat{V} \Lambda$ with Λ a real symmetric matrix. The corresponding Lagrangian is now no longer invariant but changes by a total derivative proportional to Λ . The latter is induced as a result of monodromies around the semi-classical singularities in moduli space. For finite string coupling we expect only some discrete subgroup of $SO(2, n-1)$ to be relevant [24]; Λ is then integer-valued as well.

Furthermore one can show that $F_{1\text{-loop}}(X)$ can be written as $F_{1\text{-loop}}(X) = \frac{1}{2} \hat{F}_I \hat{X}^I$. Substitution of (26) then yields at once the variation under target-space duality

$$F_{1\text{-loop}}(\tilde{X}) = F_{1\text{-loop}}(X) + \frac{1}{2} \Lambda_{IJ} \hat{X}^I \hat{X}^J. \quad (27)$$

The one-loop contribution to the function $\mathcal{F}_{1\text{-loop}} \equiv i(X^0)^{-2} F_{1\text{-loop}}(X)$ must therefore be invariant under target-space duality transformations, up to a restricted polynomial of the moduli with discrete real coefficients [19]. According to (27) $F_{1\text{-loop}}$ is

⁴Note that the following results are expressed in terms of the periods and not in terms of a new function $\hat{F}(\hat{X})$. The reason is that the \hat{X}^I are not independent and no suitable function exists. Fortunately the latter is merely a technical problem as the full Lagrangian can still be written down consistently in terms of the periods and their derivatives [23].

multivalued and the ambiguities in this function amount to the quadratic polynomial in the variables \hat{X}^I (to see this consider a transformation with $U = \mathbf{1}$, so that the \hat{X}^I remain unchanged, but $\Lambda \neq 0$).

As an explicit example one may consider toroidal compactifications of six-dimensional $N = 1$ string vacua, where we have only T and U . The transformation $T \rightarrow (aT - ib)/(icT + d)$ with integer parameters satisfying $ad - bc = 1$, then induces the following result on the one-loop correction (one can also consider a similar transformation of U),

$$\mathcal{F}_{1\text{-loop}}(T, U) \rightarrow (icT + d)^{-2}[\mathcal{F}_{1\text{-loop}}(T, U) + \Xi(T, U)], \quad (28)$$

where Ξ is a quadratic polynomial in the variables $(1, iT, iU, TU)$. Hence $\partial_T^3 \Xi = \partial_U^3 \Xi = 0$. The appearance of Ξ complicates the symmetry properties of the one-loop term, which would otherwise be a modular function of weight -2 . However, the third derivative $\partial_T^3 \mathcal{F}_{1\text{-loop}}$ transforms as a modular function of weight $+4$ under T -duality and of weight -2 under U -duality⁵. The same statement applies to $\partial_U^3 \mathcal{F}_{1\text{-loop}}$ with the modular weights interchanged. The above result was also derived in [25] for the specific case of the toroidal compactification.

The polynomial Ξ encodes the monodromies at singular points in the moduli space (for instance, at $T \approx U$) where one has an enhancement of the gauge symmetry. Knowledge of these singularities and of the asymptotic behaviour when $T \rightarrow \infty$ or $U \rightarrow \infty$, allows one to uniquely determine

$$\partial_T^3 \mathcal{F}_{1\text{-loop}}(T, U) = \frac{1}{2\pi} \frac{E_4(iT) E_4(iU) E_6(iU)}{(j(iT) - j(iU)) \eta^{24}(iU)}, \quad (29)$$

where η is Dedekind's eta-function, E_4 and E_6 are the normalized Eisenstein's modular forms of respective weights $+4$ and $+6$ and j is the modular invariant function $j = E_4^3/\eta^{24}$. A similar formula can be obtained for the third derivative with respect to U .

We refer to [19] for further details. Before closing we wish to point out that the dilaton field S is no longer invariant under target-space duality in the presence of the one-loop corrections. This can be understood from the fact that the dilaton belongs originally to a vector-tensor multiplet and is only on-shell equivalent to a vector multiplet. However, one can always redefine S such that it becomes invariant, but then it can no longer be interpreted as the scalar component of a vector multiplet [19]. Interestingly enough, these perturbative results are confirmed by explicit calculations based on 'string duality' between heterotic string compactifications on $K_3 \times T_2$ and type-II string compactifications on Calabi-Yau manifolds [26].

⁵We note that, when $F(z') = (icz + d)^\nu G(z)$ with $z' \equiv (az - ib)/(icz + d)$ and $ad - bc = 1$, we have the following relation for multiple derivatives:

$$(\partial^n F)(z') = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\nu + n)}{\Gamma(\nu + k)} (ic)^{n-k} (icz + d)^{\nu+n+k} \partial^k G(z).$$

When $n = 1 - \nu$, only the highest derivative survives on the right-hand side.

This talk is based on work done in collaboration with V. Kaplunovsky, J. Louis, D. Lüst and A. Van Proeyen. I thank B. Kleijn for useful comments on the manuscript. This work was carried out in the framework of the European Community Research Programme “Gauge theories, applied supersymmetry and quantum gravity”, with a financial contribution under contract SC1-CT92-0789.

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